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PARTITION FUNCTION IN TERMS OF THE EULER'S TOTIENT FUNCTION

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Abstract:

Merca obtained an expression involving the partition function $p(n)$ and Euler's totient function $\varphi(k)$, which can be written using a lower triangular matrix whose inversión gives the relation deduced by Alegri-Prajapati-(López-Bonilla) for $p(m)$ in terms of $\varphi(j)$.

Key Words: Partition Function, Euler Totient Function, Lower Triangular Matrix

1. Introduction:

Merca [1] deduced the expression:

$$\sum_{k=1}^{n} k\ p(n-k) = \sum_{k=1}^{n} \varphi(k)\, S_{n,k}\ , \qquad (1)$$

Involving the partition and Euler totient functions [2-5], where $S_{n,k}$ is the number of k's in all partitions of n. Similarly [1]:

$$p(n) = \sum_{k=1}^{n+1} \mu(k)\, S_{n+1,k}\ , \qquad (2)$$

With the participation of the Möbius function [2, 6, 7]. The relations (1) and (2) provide remarkable properties connecting a function of multiplicative number theory with one of additive number theory.

If we introduce the lower triangular matrix:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 2 & 1 & 0 & 0 & \cdots & 0 \\ 3 & 2 & 1 & 0 & \cdots & 0 \\ 4 & 3 & 2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n-1 & n-2 & \cdots & \cdots & 1 \end{pmatrix}, \qquad (3)$$

That is, $A_{m,m-j} = j+1,\ j = 0,1,\dots,n-1,\ m = j+1, j+2,\dots,n$, then from (1) we obtain the following explicit relation for the partition function:

$$\begin{pmatrix} p(0) \\ p(1) \\ \vdots \\ p(n-1) \end{pmatrix} = \boldsymbol{A}^{-1} \begin{pmatrix} q(1) \\ q(2) \\ \vdots \\ q(n) \end{pmatrix}, \quad q(m) := \sum_{k=1}^{m} \varphi(k)\, S_{m,k}\ ,\ 1 \le m \le n\ . \qquad (4)$$

In Sec. 2 we determine the inverse of matrix (3) and thus (4) generates to $p(n)$ in terms of $\varphi(j)$.

2. Inversion of $\boldsymbol{A}$

The inversion of (3) is immediate if we use the technique indicated in [8], hence:

$$\boldsymbol{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 1 & -2 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 \end{pmatrix}, \qquad (5)$$

Whose application in (4) gives the property:

$$p(m) = q(m+1) - 2\,q(m) + q(m-1),\ \ m = 0,1,2,\dots,\quad q(-1) = q(0) = 0, \qquad (6)$$

Which is equivalent to the following formula:

$$p(n) = \sum_{k=1}^{n+1} \varphi(k)\left(S_{n+1,k} - 2\,S_{n,k} + S_{n-1,k}\right),\qquad n \ge 0, \qquad (7)$$

Because $S_{n,m} = 0,\ n < m,\ S_{r,0} = 0,\ r \ge 0,\ S_{t+1,t} = 1,\ t \ge 2,\ S_{j,j} = 1,\ j \ge 1$; this expression (7) was deduced in [9] employing the Z-transform [10, 11].

Thus, our analysis shows that the inverse of certain lower triangular matrix allows obtain the explicit relation (7) for the partition function $p(n)$ in terms of Euler's totient.

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