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TOTALITY OF PARTS IN ALL PARTITIONS OF AN INTEGER

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Abstract:

We employ the Merca and Uchimura theorems to obtain an expression for the totality of parts in all partitions of an integer.

Key Words: Merca's Theorem, Integer Partitions, Divisor Function, Uchimura's Theorem.

1. Introduction:

Merca [1-3] deduced the following expression for an arbitrary arithmetic function $f(n)$:

$$\sum_{K=1}^{n} f(k)\, S_{n,k} = \sum_{k=1}^{n} g(k)\, p(n-k),$$
$$g(m) = \sum_{d/m} f(d), \tag{1}$$

Where $S_{n,k}$ is the number of k's in all partitions of n, and $p(m)$ is the partition function [4].

For the case $f = e$, that is, $f(k) = 1$, then $g(m)$ is the divisor function $d(m)$ [4] and thus (1) gives the totality of parts in all partitions of the integer n:

$$\sum_{k=1}^{n} S_{n,k} = \sum_{k=1}^{n} d(k)\, p(n-k). \tag{2}$$

On the other hand, if $Q_j(n)$ is the number of partitions of n into (possibly repeated) parts from among $\{1, 2, \ldots, j\}$, its generating function is given by [5]:

$$\sum_{n=0}^{\infty} Q_j(n)\, q^n = \frac{1}{(q\,;q)_j}, \tag{3}$$

Such that $\lim_{j\to\infty} Q_j(n) = p(n)$, that is:

$$\sum_{n=0}^{\infty} p(n)\, q^n = \frac{1}{(q\,;q)_\infty}. \tag{4}$$

In Sec. 2 we use a theorem of Uchimura [5, 6] to deduce a connection between (2) and the $Q_j(n)$.

2. All Partitions of an Integer and Their Totality of Parts:

The Uchimura's theorem gives the following relation [5, 6]:

$$\frac{1}{(q\,;q)_\infty} \sum_{n=0}^{\infty} d(n)\, q^n = \sum_{j=0}^{\infty} \frac{j}{(q\,;q)_j}\, q^j, \tag{5}$$

Where we can apply (3) and (4) to obtain:

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} d(j)\, p(n-j) \right) q^n = \sum_{j=0}^{\infty} j \sum_{m=0}^{\infty} Q_j(m)\, q^{j+m}$$
$$= \sum_{j=0}^{\infty} j \sum_{n=j}^{\infty} Q_j(n-j)\, q^n,$$
$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} j\, Q_j(n-j) q^n,$$

Therefore:

$$\sum_{K=1}^{n} S_{n,k} = \sum_{k=1}^{n} d(k)\, p(n-k)$$
$$= \sum_{k=1}^{n} k\, Q_k(n-k), \tag{6}$$

Is an interesting connection between the $Q_m(n)$ and the total number of parts in all partitions of an integer.

Remark:

$Q_k(n)$ also is the number of partitions of n into at most k parts [7].

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