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ON A Q-SERIES IDENTITY OF COOGAN - ONO

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Abstract:

We show an elementary deduction of certain q-series identity obtained by Coogan and Ono.

Key Words: Coogan-Ono's q-series identity, Z-transform, q-binomial series.

1. Introduction:

Coogan-Ono [1, 2] obtained the q-series identity:

$$\sum_{n=0}^{\infty} \frac{(z\,;q)_n}{(-z\,q\,;q)_n} z^n = (1+z)\sum_{n=0}^{\infty} (-1)^n\, q^{n^2} z^{2n}; \tag{1}$$

Here we exhibit an elementary approach to motivate (1). It is possible to generalize this identity, in fact [1]:

$$\sum_{n=0}^{\infty} \frac{(z\,;q)_n}{(-zq\,;q)_n} t^n = \frac{1}{1-t}\sum_{n=0}^{\infty} \frac{(z\,;q)_n(-t\,;q)_n}{(-zq\,;q)_n\,(tq\,;q)_n}(1-z\,t\,q^{2n})(-1)^n\, z^n\, t^n\, q^{n^2}, \tag{2}$$

Which reproduces (1) if $t = z$.

2. Z-Transform:

We wish to write the left side of (1) in powers of q and z, then:

$$\sum_{n=0}^{\infty} \frac{(z\,;q)_n}{(-zq\,;q)_n} z^n = \sum_{n=0}^{\infty} b_n\, q^{h(n)} z^n, \tag{3}$$

And we must determine b_n and $h(n)$. From (3) when $q \to 1$:

$$\sum_{n=0}^{\infty} b_n\, z^n = \sum_{n=0}^{\infty} \left(\frac{z\,(1-z)}{1+z}\right)^n = \frac{z+1}{z^2+1}, \tag{4}$$

Therefore, the Z-transform of the sequence $\{\, b_n \,\}$ is given by $Z\{b_n\} = \frac{z\,(z+1)}{z^2+1}$, and its inversion is immediate:

$$\{b_n\} = \{1, 1, -1, -1, 1, 1, -1, -1, \dots\}, \quad n = 0, 1, 2, \dots \tag{5}$$

Which generates the expansion:

$$\sum_{n=0}^{\infty} b_n\, z^n = 1 + z - z^2 - z^3 + z^4 + z^5 - z^6 - z^7 + z^8 + z^9 - \cdots, \tag{6}$$

$$= (1+z)(1 - z^2 + z^4 - z^6 + z^8 - \cdots) = (1+z)\sum_{n=0}^{\infty}(-1)^n\, z^{2n}.$$

The expression (6) suggests that instead of (3) we consider a relation with the following structure:

$$\sum_{n=0}^{\infty} \frac{(z\,;q)_n}{(-zq\,;q)_n} z^n = (1+z)\sum_{n=0}^{\infty} (-1)^n\, q^{f(n)} z^{2n}. \tag{7}$$

3. q-Series Identity:

We know the q-series [3]:

$$(z\,;q)_n = \sum_{k=0}^{\infty} a(n,k)\, z^k, \qquad a(n,k) = \binom{n}{k}_q (-1)^k q^{k(k-1)/2}, \tag{8}$$

$$\frac{1}{(-zq\,;\,q)_n} = \sum_{k=0}^{\infty} b(n,k)\, z^k, \qquad b(n,k) = \binom{n+k-1}{k}_q (-1)^k\, q^k,$$

Hence:

$$\frac{(z\,;q)_n}{(-zq\,;q)_n} = \sum_{r=0}^{\infty} c(n,r)\, z^r, \qquad c(n,r) = \sum_{j=0}^{r} a(n,j)\, b(n,r-j), \tag{9}$$

Therefore:

$$c(m,0) = 1, \quad c(0,m) = 0, \ \ m = 0, \dots, 4, \dots, \quad c(1,1) = -(1+q), \ \ c(1,2) = q(1+q),$$
$$c(1,3) = -q^2(1+q), \ \ c(2,1) = -(1+q)^2, \quad c(3,1) = -(1+q+q^2)(1+q), \tag{10}$$
$$c(2,2) = q^2(1+q+q^2)(1+q), \dots$$

Besides, from (7) and (9):

$$\sum_{n=0}^{\infty} \frac{(z\,;q)_n}{(-zq\,;q)_n} z^n = \sum_{m=0}^{\infty} Q_m\, z^m, \qquad Q_m = \sum_{j=0}^{m} c(j, m-j), \tag{11}$$

$$= \sum_{n=0}^{\infty} (-1)^n\, q^{f(n)}(z^{2n} + z^{2n+1}),$$

That is:

$$Q_0 = Q_1 = q^{f(0)} = 1, \ \ Q_2 = Q_3 = -q^{f(1)} = -q, \ \ Q_4 = Q_5 = q^{f(2)} = q^4, \ \ Q_6 = Q_7 = -q^{f(3)} = -q^9, \dots$$

Which suggests that $f(n) = n^2$, then (7) implies the q-series identity (1), q.e.d.

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