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HOMOTOPY OF PATH IN TOPOLOGY

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Abstract

In this thesis we shall introduce the most basic of the object in algebraic topology, the fundamental group. For this purpose we shall define the notion of homotopy of paths in a topological space X and show that this is an equivalence relation. We then fix a point x0 ∈ X in the topological space and look at the set of all equivalence classes of loops starting and ending at x0. The sets are endowed with as a binary operation that term turns it into a group known of the fundamental group π1(X, x0). Always besides of the most basic object in algebraic topology, it is of paramount importance in low dimensional topology. A detailed study of this group will occupy the rest of part I of this course. In this thesis we should focus only on the most elementary results. Algebraic topology is mostly about bonding invariants for topological spaces. We will primarily be interested in topological invariants that are invariant under certain kinds of smooth deformations called homotopy. In general, we will be able to associate an algebraic object (group, ring, module, etc.) to a topological space. The fundamental group is the simplest, in some ways, and the most di¢ cult in others.

Key Words: Topology, Homotopy & Path

Introduction:

One of the basic problems of topology is to determine whether two given topological spaces are homeomorphic or not. There is no method for solving this problem in general, but techniques do exist that apply in particular cases. Shows that two spaces are homeomorphic as a matter of constructing a continuous mapping from one to the other having a continuous inverse, and constructing continuous functions is a problem that we have developed techniques to handle.

Shows that two spaces aren't homeomorphic are a different matter. For that, one must show a continuous function with continuous inverse *does not exist.* If one can find some topological property that holds for one space but not for the other, then the problem is solved-the spaces cannot be homeomorphic. The closed interval [0, 1] cannot be homeomorphic to the open interval (0,1).For instance, because the first space is compact and the second one is not. And the real line $\mathbb{R}$ homeomorphic to the "long line" L, because $\mathbb{R}$ be homeomorphic to the plane $\mathbb{R}^2$ leaves a connected space remaining and deleting a point from $\mathbb{R}$ does not.But the topological properties we have studied up to now do not carry us very far in solving the problem.

We must introduce a most natural property called simple connectedness. The property of simply connectedness will distinguish between $\mathbb{R}^2$ and $\mathbb{R}^3$; deleting a point from $\mathbb{R}^3$ leaves a simply connected space remaining, but deleting a point from $\mathbb{R}^2$ does not. It will ailso distinguish between S^2 and torus T.But it will not distinguish between T and T#T; neither of them is simply connected. It involves a certain group that is called fundamental group of the space. Two spaces that are homeomorphic have the fundamental groups that are isomorphic. Thus the proof that S^3 and T^3 are not homeomorphic can be rephrased by that the fundamental group of T is not. For example, to show that T and T#T are not homeomorphic; it turns out that T has an abelian fundamental group and T#T does not.

Preliminaries Definition:

A Topology on a set x is a collection Ĵ of subsets of X having the following properties,

- ∅, X ∈ Ĵ.
- The union of the elements of any finite sub collection of Ĵ is in Ĵ.
- The intersection of the elements of any finite sub collection of Ĵ is in Ĵ .

A set X for which a topology Ĵ has been specified together is called a **Topological space.**

Example:

Let X=$\{a,b,c\}$

$\hat{J}_1$ =$\{\emptyset,\{a\},\{b\},\{a,b\},X\}$ is a topology for X.

$\hat{J}_2$=$\{\emptyset,\{a\},\{b\},X\}$ is not a topology for X.

Definition: Let X b e a set then Ĵ=$\{\rho(X)\}$ is always a topology on X and it is called the Discrete Topology on X.

Example:

Write down the discrete topology for the set X=$\{1,2,7\}$

ρ (X)= $\{\emptyset,X,\{1\},\{2\},\{7\},\{1,2\},\{1,7\},\{1,2,7\}\}$

ρ(X)= 8.

Definition: Let X be a set then $\hat{J}=\{\emptyset, X\}$ is always a topology on X is called the Indiscrete Topology on X.
Example: Let X be any finite set. Then $\hat{J}=\{\emptyset, X\}$ is an indiscrete topology.
Definition: Let X and Y be topological spaces. A space f:$X \rightarrow Y$ is said to be Continuous. If for each open subset V of Y, the set f^{-1} (V) is an open subsets of X.
Example: Let f(x) =y_0 ∀ x∈ X . Let V be an open set in Y.
To Prove:

f^{-1}(V) is open Y ;
f : X → Y ,If $y_0 \in V$ then f^{-1}(V)=X^{open}
∴ f^{-1} (v) is open
f is continuous.

Definition: An Equivalence Relation on a set A is a relation c on A have there is a following three properties,

- (Reflexivity) : if x c x for every x in A.
- (Symmetry): if x c y then y c x.
- (Transitivity) if x c y and y c x then x c z.

Pasting Lemma:

Let X=AUB where A and B are closed in X.
Let f : A→ Y and g : B→ Y be continued.

If f (x) = g (x) ∀ x ∈ A∩ B then f and g combine to give a continuous function h : X→ Y is defined by setting

$$h(x)=\begin{cases} f(x) \ if \ x \in A \\ g(x) \ if \ x \in B \end{cases}$$

Definition: A topological space is a set X together with a collection of subsets of X called Open Sets of X such that ∅ and X are open sets. (i.e) All the elements of J is open sets X. Arbitrary union of open set is open.
Definition: Let X be a space, let x_1, $x_2 \in X$ by a Path in X from x_1 to x_2 is a continuous map, f : [a,b] → X such that f(a) = x_1 ;f(b) = x_2 , where [a,b] is any closed interval of a real line .
Definition: A space X is said to be Path Connected. If every pair of points in X there exists a path between them.
Definition: A subset A of a topological space X is said to be closed set if X-A is open.
Example: [a,∞) is closed in ℝ.
Proof:
R- [a,∞) = (-∞,a)
R- [a,∞) = open set ∴ [a,∞) is closed.
Definition: A map f from a group G into a group G' (i.e) f : G→ G′ is called a Homomorphism.
Example:

If f : G → G' is defined by f (x) = e'
For , f (xy) = e'

Homotopy of Paths:
Introduction: Before introducing the fundamental group of a space X, we consider paths on X and an equivalence relation called path homotopy between them.
Definition: If f and f ′ are continuous map of the space X to the space Y then it is said f is homotopic to f ′, if there is a continuous map

F : X × I → Y such that for each x.(here I = [0,1])
F(x , 0) = f (x) and F (x ,1) = f ′(x).

The map F is called a homotopy between f and f ′. If f is homotopic to f ′.
It is denoted by f ≈ f' .
Definition: If f ≈ f ′ (f is homotopic to f ′) and f ′ is a constant map then it is called an nulhomotopic.
Remark: A homotopic as a continuous one parameter family of maps from X to Y. The parameter " t " representing time. The homotopy F represents a continuous " deforming " of the map f to the map f ′, as t goes from 0 to 1.
Definition: If f is a path in X, f : [0,1]→ X is a continuous map such that, f (0) = x_0 and f(1) = x_1 , then f is a path in X from x_0 to x_1. if x_0 is the initial point, And if x_1 is the final point of the path f.
Note: For convenience use the interval I = [0,1] as domain for all paths. If f and f ′ are two paths in X, there is a stronger relation between them than mere homotopy.
The Fundamental Group:
Introduction: The set of path - homotopy classes of paths in a space X does not form a group under the operation ∗ because the product of two path – homotopy classes is not always defined. But suppose we pick out a point x_0 of X to serve as a " base point " and restrict ourselves to those points that begin and end at x_0.The set of these path – homotopy classes does form a group ∗ .it will be called an fundamental group of X.
In this section, we shall study the fundamental group and derive some of its properties .in particular we

shall show that the group is a topological invariant of the space X, fact of that is too crucial importance in using are study homeomorphism problems.

Definition: Suppose G and G ′ are groups, a homomorphism f : G → G' is a map such that, f (x.y) = f (x).f (y) for all the sets of x, y ; it is automatically satisfies the equation,

$$f(e) = e' \text{ and } f(x^{-1}) = f(x)^{-1}$$

Where e and e ′ are the identities of G and G ′.respectively, and the exponent -1 denotes the inverse.

Definition: The kernel of f is the set $f^{-1}(e')$;it is a group of G. the image of f , similarly, is a subgroup of G ′ **.**

Definition: A homomorphism f is called one to one if its one to one or equivalently, hence if the kernel of f consists of e alone. It is called an epimorphism if it is surjective. And it is called an isomorphism if it is bijective.

Definition: Suppose G is as of a group and H is of a subgroup G. Let xH denote the set of all products xh , for h ∈ H it is called a left coset of H in G. The collection of all cosets forms a partition of G, similarly, the collection of all right cosets Hx of H in G forms a partition of G.

Definition: A H is a normal subgroup of G. if $x.h.x^{-1} \in H$ for each x ∈ G and for each h ∈ H .in this case, x H = H x for each x, so that two partitions of G are the same.

Definition: This partition by G/ H ; if one defines (x H). (y H) = (x.y)H one obtains a well – defined operation on G/H that makes it a group. This group is called the quotient to the G by H. The map f : G → G/H carrying x to xH is an epimorphism with kernel H. conversely , if f : G → G' is an epimorphism then its kernel N is a normal subgroup of G and induces an isomorphism G/ N → G' that carriers xN of f (x) for each x ∈ G**.**

Note: If the subgroup H of G is not normal , it is convenient to use the symbol G / H .it will use to denote the collection of right cosets of H in G.

Covering Spaces

Introduction : Any convex subspace of $\mathbb{R}^n$ has a non-trivial fundamental group to the task of computing some fundamental groups that is non-trivial .one of the most useful tools for this purpose is the notion of covering space, we introduce this section. Covering spaces was also important in the study Riemann surfaces at complex manifolds.

Definition: Let p : E→ B be a as continuous surjective in bellowed map. The open set in the queue U of B is said to be evenly covered by p. if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that, For every α, the restriction of p to V_α are a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into slices.

Remark: If U is an open set that is evenly covered by p. we often picture the set $p^{-1}(U)$ as a " stack of pancakes " each having the same size and shape as U ,floating in the air above U; the map p squashes them all down onto U.

Theorem: If the function p : E → B be a covering map . If B_0 is a subspace of B, and if $E_0 = p^{-1}(B_0)$, then map $p_0 : E_0 \rightarrow B_0$ obtained by restricting p is a covering map.

Proof: Given, $b_0 \in B_0$

Let U be an open set in B containing b_0 ;

(i .e.,) evenly covered by p;

let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices .

Then $U \cap B_0$ is a neighbourhood of b_0 in B_0 and the sets $V_\alpha \cap E_0$ are disjoint open sets in E_0 whose union is $p^{-1}(U \cap B_0)$, and each is mapped homeomorphically onto $U \cap B_0$ by p.

Definition: If the function p : E → B be a covering map. let B be connected. If $p^{-1}(b_0)$ has k-elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for all the kinds b ∈ B. E is called k- fold always covering of B.

Theorem 4.3: If p : E → B and p′: E′ → B′ are covering maps, then $p \times p' : E \times E' \rightarrow B \times B'$ is a covering map.

Conclusion:

I gave some basic concept of the algebraic topology .Algebraic topology is derived from combinatorial topology and it is a models in topological entities and relationships as a algebraic structures such as groups (or) a rings. This project deals with fundamental concepts which are more useful for the development of the theory on homotopy of paths.

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