International Journal of Advanced Trends in Engineering and Technology (IJATET)
Impact Factor: 5.965, ISSN (Online): 2456 - 4664
(www.dvpublication.com) Volume 4, Issue 2, 2019

AN INTRODUCTION ON NORMED SPACE IN FUNCTION ANALYSIS

P. Muralitharan* & M. Rajamohan**
* Research Scholar, Department of Mathematics, PRIST University, Vallam, Thanjavur, Tamilnadu
** Assistant Professor, Department of Mathematics, PRIST University, Vallam, Thanjavur, Tamilnadu

Cite This Article: P. Muralitharan & M. Rajamohan, "An Introduction on Normed Space in Function Analysis", International Journal of Advanced Trends in Engineering and Technology, Volume 4, Issue 2, Page Number 15-19, 2019.

Abstract:

Functional analysis was born in the early years of this decade as part of a huge trend toward abstractions - Some authors have reviewed as "arithmetization" of analysis. This same view towards "axiomatics" contributed for the foundations of abstract linear algebra's, modern geometric equations, and topology. Functional analysis always a very broad field, encompassing much of the modern analysis. In the way, it would be difficult to bring a simple definition as what functional analysis means lively. Either discussed its present meaning, we would concentrate on its foundations with settle for an all too brief description of modern trends. We would discuss an lots of episodes from early history of "abstract analysis," especially who those related to an improvement of vector spaces and other "abstract spaces." of unique interest to us will be the movement from the specific to the generic in mathematics; as one example of this, you may be surprised to learn that the practice of referring to functions by "name," writing a single letter f, say, rather than the referring to its values f(x), only became common in our own century. As particular, we would discusses the work of Fredholm and Hilbert of integral equations with operator theory, these work of Volterra and Hadamard for the problem of moments, the work of Lévesque, Fr´echet, and Riesz on abstract spaces, and the work of Helly, Hahn, and Banach on the notion of duality. Additionally, we would present a few samples which illustrating of functional analytic viewpoint.

Introduction:

Introducing the concept of normed space which is fundamental to the development of the theory of banach space. The notion of normed space can be thought of as a generalization of the n - dimensional unitrary space with the Euclidean length given by $(\sum_{k=}^{n} |z_k \,|)$. The norm ||u||, which is assigned to each element u in a norlned linear space V, will then be used to define, a metric and hence the convergence $u_n \rightarrow u$ as $n\rightarrow \infty$ in V. by means of the equivalent condition $||u_n - u|| \rightarrow 0$ as $n\rightarrow \infty$.

1. Basic Definitions:

Definition 1.1: Let X bee need space. A norm ||.|| on X is a function from X to R^{+}, satisfied

- $||x|| \geq 0$ and $||x|| = 0 <=> x = 0$
- $||\propto x|| = |\propto|||x||\forall x \in X$ and scalar a
- $||x+y|| \leq ||x|| + ||y||$

A vector space with a nonn is called a normed space.

Definition 1.2: A banach space is a formed linear space that is a complete metric space with respect for the metric derived from its norm.

- $||x|| \geq 0 \& \big||x|\big| = 0 <=> x = 0$
- $||x+y|| \leq ||x|| + ||y||$
- $||\propto x|| = |\propto ||x||| \;\forall x \in X$ and $\propto \in F$

Definition 1.3: Let X be non-empty set. A metric on X is a real valued function satisfying the following conditions

$d(x,y)$ and $d(x,y) = 0 \;<=> x = y$, $\forall$ x, y $\in$x
$d(x,y)$ $d(y,x)$ for every x, y $\in$ X
$d(x,y) \leq d(x,z) + d(z,y)$ for any x, y, z $\in$ X
$d(x,y)$ is called the distance belongs x and y.

It is a finite non - negative real number. A non-empty set X together with a metric d defined on X is called a metric space.

Definition 1.4: A vector space (or) linear space over the field F is a set X with the operation called vector addition defined on $X \times X$ to X given by (x, y) = x + y and an operation called scalar multiplication defined on $F \times x \rightarrow X$ given by (a, x) = a(x) satisfying the following condition. For all x,y,z$\in$X and $\propto, \beta \in F$

- (x+y) + z =(y+z)
- x+y = y+x
- There exist an element 0$\in$X called a zero vector such that x + 0 = x
- For each x G X ,there exist an element -x $\in$ X

called the addition inverse of x such that x + (-x) = 0

$$\propto (x+y) = \propto x + \propto y$$
$$(\propto + \beta)x = \propto x + \beta x$$
$$\propto (\beta x) = \propto \beta(x)$$
$$1.x=x$$

Definition 1.5: A non - empty subset W of a linear space over K is said to be a linear subspace if the following two conditions are satisfied

- x+y ∈W,∀ x, y ∈ W
- $\propto x \in W, \forall \propto \in k$ *and* ∀x∈ W

2. Properties of Norm:

Definition 2.1: The concept of norm was introduced in order to give a method for measuring the magnitude of a vector.

Example: x =(-1,-2,-3,-7,-11)is in R^5. The ||x||=11 is the vector norm. Which is the length of the largest co-ordinates.

Definition 2.2:

Euclidean Norm: The Euclidean norm c^n is will be defined by $||z||_2$ = $(\sum_{k=1}^{n} | z_k^2|)^2$,z=$(z_1, z_2, \ldots \ldots \ldots . z_n)$ ∈C^norigin .

Example: If n = 1 we have R^1 = R and C^1 = C. So that x ∈R . ||x||=|x|. The absolute value of the real number x, and that Z=x+iy∈C

$$||Z||=|Z|=\sqrt{x^2+y^2}$$

Where are the modulus of the complex number Z. The length of the vector emanating from (0,0) to (x, y) ∈ R^2≅C.

Definition 2.3:

P - Norm: P -norm is defined by, $||z||_p$ = ($\sum_{k=1}^{n} |z|p)$if 1 ≤ p <∞

P= 2 this corresponds to the Euclidean norm in C^n. The norm based on the length of the largest co- ordinate corresponds to p= ∞,which is given by,

$$||Z|| = \max |Z_k|$$

$1 \leq k \leq n$

Definition 2.4:

Simple Norm: A simple norm on R^2.(or) a l- norm on R^2 defined by,||x||= $|x_1|+|x_2|$,x=(x_1+x2)∈R^2

Definition 2.5:

Elliptical Norms: Elliptical norm $||.||_e$ is given by, ||x||e = $\sqrt{\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}}$, x = (x_1,x_2) ∈ R^2 for some fixed a, b > 0. The unit ball on the normed space (R^2,||.||e)is then given by ,

$$\{ x\in R2 , || x||e < 1\} = \{(x1,x2)\in R2 ; \frac{x_1^2}{a^2}+\frac{x_2^2}{b^2} < 1\}$$

(x GR2 'llxlle< l}

3. Convexity and Completeness:

Definition 3.1: A non - empty subset S of a vector space V is said to be convex, if $\lambda x_1 + (1-\lambda)x_2 \in S$ .whenever x_l , x_2∈ S, and 0 ≤ λ < 1. This means that if given two arbitrary points in a convex set, then the line segment joining them is also in the set {λ x_1 +(1-λ) x_2 ,λ∈ R }

Definition 3.2: A metric space is called a complete space iff every Cauchy sequence of points in it converges to a point in the space.

Definition 3.3:

Frechet Metric: Let X be the set of all infinite sequence of complex numbers not necessarily convergent, not even bounded . Let {k_n } be an arbitrary fixed sequence of positive real numbers such that the sum $\sum_{x=1}^{\infty} x_0$ Conveges.

$$\text{(for instance, kn} = 2^{-1}, 3^{-2}\text{(or)} \frac{1}{x!}\text{etc)}$$

For Z={z_n}n ≥ 1 and w ={w_n}n≥ 1in X. defined d by d(z,w)=$\sum_{x=1}^{\infty} x_x \frac{|x_1-x_{1n}|}{1+|x_1-x_0|}$

The series defined d (z , w) converges,

$$\text{Since, } k_n\frac{|x_1-x_1|}{1+|x_1-x_0|} \leq k_n$$

And that kn$\sum_{x=1}^{\infty} x_x$ converges. The triangle inequality d(x,y) ≤ d(x,z) +d(z,y) is immediate from the inequality

$$\cdot \frac{|x_1-x_1|}{1+|x_1-x_0|} \leq + \frac{|x|}{1+|x|}$$

Since, d(z,w) is bounded. Thes metric is called frechet metric for X

Proposition 3.4:

The open ball B (a,R) = { x ;||x – a|| < R} in a normed space is convex. In particular, the function f defined by f (x)=||x|| is convex.

Proof:

The open ball B(a,R)={x;||x-a|| <R} in a normed space is convex. If x,y∈ B (a,R) and λ∈(0,1), then convex define,

$\leq \|\lambda x - \lambda a - \lambda a - \lambda a + (1-\lambda)y - a\| = \|\lambda x - \lambda a + (1-\lambda)y + \lambda a - a\|$
$= \| \lambda (x-a) + (1-\lambda)y + \lambda a - a\| \leq \| \lambda(x-a) \| + \|(1-\lambda)(y-a)\|$
$\leq |\lambda|\|x-a\| + |1-\lambda|\|y-a\| \leq \lambda\|x - a\| + (1-\lambda)\|y-a\|$
$< \lambda R + (1-\lambda)R \leq \lambda R + R - \lambda R$
$\|\lambda x = (1-\lambda)y - a\| = R$

Example: Frechetmetric d on the space of all sequence of complex numbers.

$$D(z,w) = \sum_{x=1}^{\infty} 3^{-x} \frac{|x_1 - x_1|}{1+|x_1-x_0|}$$

Proof:

Let z = {p, 0,p, 0, ………..}
W = { q,q,…………….}
where p and q are fixed + ve real numbers with q∈(0,3)

Then $d(z, 0) = \sum_{x=1}^{\infty} 3^{-x} \frac{|x_1 - x_1|}{1+|x_1-x_0|}$

$= \sum_{x=1}^{\infty} 3^{-x} \frac{|x_1 - x_1|}{1+|x_1-x_0|}$

$= \frac{1}{3}\frac{|x_1 - x_1|}{1+|x_1-x_0|} + \frac{1}{3^2}\frac{|x_1 - x_1|}{1+|x_1-x_0|} + \frac{1}{3^3}\frac{|x_1 - x_1|}{1+|x_1-x_0|} + \ldots\ldots = \frac{1}{3}\frac{|x_1 - x_1|}{1+|x_1-x_0|} + \frac{1}{3^2}\frac{|x_1 - x_1|}{1+|x_1-x_0|} + \frac{1}{3^3}\frac{|x_1 - x_1|}{1+|x_1-x_0|} + \ldots\ldots$

$= \frac{1}{3}\frac{|x_1 - x_1|}{1+|x_1-x_0|} + \frac{1}{3^2}\frac{|x_1 - x_1|}{1+|x_1-x_0|} + \frac{1}{3^3}\frac{|x_1 - x_1|}{1+|x_1-x_0|} + \ldots\ldots = \frac{x}{1+x}\left(\frac{1}{3} + \frac{1}{3^3} + \frac{1}{3^5} + \ldots\right) = \frac{p}{1+p}\sum_{n=1}^{\infty}\left(\frac{1}{3^{2n-1}}\right)$

$d(w, 0) = \sum_{x=1}^{\infty} 3^{-x} \frac{|x_1 - x_1|}{1+|x_1-x_0|} = \sum_{x=1}^{\infty} 3^{-x} \frac{|x_1 - x_1|}{1+|x_1-x_0|} = \sum_{x=1}^{\infty} 3^{-x} \frac{|x_1 - x_1|}{1+|x_1-x_0|} = \sum_{x=1}^{\infty} 3^{-x} \frac{|x_1 - x_1|}{1+|x_1-x_0|}$

$= \frac{1}{3}\frac{|x_1 - x_1|}{1+|x_1-x_0|} + \frac{1}{3^2}\frac{|x_1 - x_1|}{1+|x_1-x_0|} + \frac{1}{3^3}\frac{|x_1 - x_1|}{1+|x_1-x_0|} + \ldots\ldots = \frac{1}{3}\frac{|x_1 - x_1|}{1+|x_1-x_0|} + \frac{1}{3^2}\frac{|x_1 - x_1|}{1+|x_1-x_0|} + \frac{1}{3^3}\frac{|x_1 - x_1|}{1+|x_1-x_0|} + \ldots\ldots$

$= \sum_{x=1}^{\infty} 3^{-x} \frac{|x_1 - x_1|}{1+|x_1-x_0|}$

But since $(1-x)^{-1} = \sum_{x=1}^{\infty} 3^{-x}$

$(1-x)^{-1} = 1 + x + x^2 + \ldots$

$\sum_{x=1}^{\infty} \frac{1}{3^{2n-1}} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots$

$\sum_{x=1}^{\infty} \frac{1}{3^n} = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots$

$\sum_{x=1}^{\infty} \frac{1}{3^{2n}} = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^6} + \ldots$

$\sum_{x=1}^{\infty} \frac{1}{3^{2n}} = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^6} + \ldots$

$\sum_{x=0}^{\infty} \frac{1}{3^n} - \sum_{i=0}^{\infty} \frac{1}{3^2} = \left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\right) - \left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \ldots\right) = \frac{1}{3} + \frac{1}{3^3} + \frac{1}{3^5} + \ldots$

$\sum_{x=0}^{\infty} \frac{1}{3^n} - \sum_{n=0}^{\infty} \frac{1}{3^{2n}} = \sum_{n=1}^{\infty} \frac{1}{3^{2n-1}}$

$\sum_{x=0}^{\infty} \frac{1}{3^{2n}} = 1 + \frac{1}{3^2} + \left(\frac{1}{3^2}\right)^2 + \left(\frac{1}{3^2}\right)^3 + \ldots = 1 + \frac{1}{3^2} + \left(\frac{1}{3^2}\right)^2 + \left(\frac{1}{3^2}\right)^3 + \ldots$

$\left(1 - \frac{1}{3^2}\right)^{-1} = \sum_{n=0}^{\infty} \frac{1}{3^{2n}}$

$\left(1 - \frac{1}{9}\right)^{-1} = \sum_{n=0}^{\infty} \frac{1}{3^{2n}}$

$\left(\frac{8}{9}\right) - 1 = \sum_{n=0}^{\infty} \frac{1}{3^{2n}}$

$\sum_{i=0}^{\infty} \frac{1}{3^{2n}} = \frac{9}{8}$

Since, $(1-x)^{-1} = 1 + x + x^2 + \ldots$

$\left(1 - \frac{1}{3}\right) - 1 = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots$

$\sum_{i=0}^{\infty} \frac{1}{3^n} \left(\frac{2}{3}\right)^{-1}$

$= \sum_{i=0}^{\infty} \frac{1}{3^n}$

$\sum_{i=0}^{\infty} \frac{1}{3^2} = \frac{3}{2}$

$\sum_{i=0}^{\infty} \frac{1}{3^2} = \sum_{i=0}^{\infty} \frac{1}{3^2} - \sum_{i=0}^{\infty} \frac{1}{3^2} = \frac{3}{2} - \frac{9}{8} = \frac{3}{8}$

$d(z, 0) = \frac{3}{8}\left(\frac{x}{1+x}\right)$

$d(w, 0) = \frac{x}{1+x}\sum_{i=0}^{\infty} \frac{1}{3^2}$

$\sum_{i=0}^{\infty} \frac{1}{3^2} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots$

$\left(1 - \frac{1}{3}\right)^{-1} = 1 + \frac{1}{3} + \frac{1}{3^2} + \ldots$

$\left(\frac{3-1}{3}\right)^{-1} - 1 = \frac{1}{3} + \frac{1}{3^2} + \ldots$

$(\frac{2}{3})^{-1}$ -1 $=\frac{1}{3}+\frac{1}{3^2}+\ldots..$
$\frac{3}{2}$ -1 = $\sum_{i=0}^{\infty}\frac{1}{3^2}$
$\frac{1}{2}$ = $\sum_{i=0}^{\infty}\frac{1}{3^2}$
d (w,0) = $\frac{1}{2}$ $(\frac{x}{1+x})$
Thus z, w ∈ B [0;$\frac{x}{2(1+x)}$] when ever
$\frac{3x}{8(1+x)}=\frac{x}{2(1+x)}$
That is 3p (2(1+q)) = q(8(1+p))
3p(2+2q) = q(8+8p)
6p+6pq= 8q+8pq
6p+6pq-8pq=8q
P(6+6q-8q) = 8
P = $\frac{8i}{6-2i}$
P = $\frac{8i}{2(3-i)}$
P = $\frac{4i}{3-i}$
$\frac{3i}{8(1+i)}=\frac{i}{2(1+i)}$
3p(2(1+q))= q (8(1+p))
3p(2+2p) = q(8+8p)
6p+6pq = 8q+8pq
6p = 8q+8pq-6pq
6p = q(8+8p-6p)
$\frac{6i}{8+8i-6i}$ = q
$\frac{6i}{8+2i}$ = q
q = $\frac{3i}{4+i}$
However, λZ+(1-λ)W∉B[0,f] for all λ∈(0,1) with
f = $\frac{i}{2(1+i)}$
If we let g = λZ+(1-λ)W = $\{g_n\}n\geq 1$,then we have

It is a simple calculation to verify that

$$d(\varsigma,0) > \frac{i}{2(1+i)} \text{ for all } \lambda \in (0, 1)$$

Which show that ς i B [0; $\frac{i}{2(1+i)}$]. in fact, in the special value

$$p = 2,\ q = 1$$

one can quickly obtain that ,

$$d(\varsigma,0) = \frac{1-i}{8(2-i)} + \frac{3(1+i)}{8(2+i)} > \frac{1}{4} <=> \lambda(1-\lambda) > 0$$

which show that λz + (1 – λ) w i B[0,$\frac{1}{4}$] .whenever z ,w B(0,$\frac{1}{4}$). To complete the proof for other values of p>0 and q∈ (0,3), we must verify that

$$f(\lambda) > f(0) \text{ for all } \lambda \in (0,1) \text{ since } p = \frac{4i}{3-i} > 0,$$

we' ve
$<=>\frac{1}{1+(1-i)i}+\frac{3}{1+i(i/(3-i)}<4$
$<=>\{\frac{1}{1+(1-i)i}-1\}+3\{\frac{3-i}{3-i+i}-1\}<0$
$<=>\lambda q\{\frac{1}{1+(1-i)i}-\frac{3}{3-i+i}\}<0$
$<=>\lambda(1-\lambda)>0$, since $\lambda >$ o and q ∈ (0,3)
A similar conclusion may be draw with 2^{-n} in place of 3^n in the metric expression of d(z ,w)

Conclusion:

In this project, I have about discussed basic definitions of normed space. And discussed the Properties of norm and based theorem.1 have discussed the convergence and metric space with geometry of norms. Then I have also discussed convexity and convergent sequence in normed space is a Cauchy sequence and based theorem.

References:

1. G. Bachman, and L. Narici, functional analysis, Academic press, 1966.

2. M.M. Day, Normed linear space, Springer - Verlag, 1973.
3. P. R Halmos, Finite dimensional vector space, Springer -Verlag, 1958.
4. R. T. Rockafellar, Convex analysis, Princeton university press, 1970
5. W. Rudin, Functional analysis, McGraw- Hill book company Inc., 1973. 43