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LEBESGUE INTEGRALS THROUGH VALUES AT COUNTABLE NUMBER OF POINTS

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Abstract:

A decade ago, a Frenchman mathematician Henri Lebesgue identifies that the Riemann Integral doesn't work well on unbounded functions. It moves forward him to think of another approach to do the integration, which is called Lebesgue Integral. This paper will briefly talk about the inadequacy of the Riemann integral, and introduce a more comprehensive definition of integration, the Lebesgue integral. There are also some discussions on Lebesgue measure, which establish the Lebesgue integral. Some exemplariness, like Fσset, Gδ set also Cantor function will also be mentioned.

Introduction:

In the mathematics world, a countable set is a universal set with the same cardinality as some subset of the set of natural numbers. A countable set is either a finite set or a countable infinite set. Whether finite or infinite at in the way, the values of a countable set can always be counted one at a time and, although the counting may never finish, each element of the set is associated with a unique natural number. A set S is countable if there exists an injective function f from S to the natural numbers N = {0, 1, 2, 3, ...}.If such an f can be found that is also surjective (and therefore bijective), then S is called countably infinite. In another way, a set is countably infinite if it has a one-to-one correspondence with the natural number set, N.As skins above, this terminology is not a universal one. Some reviewers are used countable to mean what is here called countably infinite, and also do not include finite sets.

Preliminaries:

Bounded Function: A function f: A R is said to be bounded on A, if there exists a real number M>0 such that, |f(x)| M, x A.

Riemann Integration: Let [a, b] be a given closed interval. By a partition of [a, b] , we mean a finite set P of pointsx0,x1,x2…….xn, Where To, Cl,X2, ...cn , where, a [x0,x1],[x1,x2],…….[xi-1,xi],…[xn-1,xn] are the sub-intervals of [a, b]. We shall use the same symbol to denote the i^{th} sub-interval [xi-1,xi] as its sub-length xi-xi-1.This, xi=xi-xi- 1,(i=1,2,3,…..n). Let the f be a bounded real-valued function on [a, b]. F is bounded on each sub-interval corresponding to each partition P. Let Mi, mi be the bounds (supremum and infimum) off in the two sums,

U (p, f) =M1M2……+Mn

L (P, f) =ixi=m1+m2x2+……+mnxn

Are respectively called the upper and lower sums of f corresponding to the partition P. If M, m are the bounds of f in [a, b], we have

mmiMi M

Putting i = 1, 2, 3, ...n and adding all the inequalities we get

m (b-a)L(P,f)U(p,f)M(b-a),ba. (0.1)

Now each partition gives rise to a pair of sums, the upper and lower sums. By considering partitions of [a, b], we get a set U of upper sums and a set L of Lower sums. The inequality (0.1) shows that both these sets are bounded and so each set has the supremum and infimum.

Upper Integral: The uniform of the set of upper sums is called the Upper integral and it is denoted by,

(x)dx=inf U or inf{U(P,f):P is a partition of [a,b]}.

Lower Integral: The supremum of the set of lower sums is called the Lower integral and it is denoted by

(x)dx=sup L or sup{L(p,f):P is a partition of [a,b]}.

Upper and Lower integrals may or may not be equal.

Riemann Integral: A bounded function f is said to be Riemann integrable (or simply integrable) over [a, b] if it's Upper and Lower integrals are equal and the common value of Upper and Lower integrals is called the Riemann integral f f on [a, b], is denoted by: (x)dx.The fact that f is integrable over [a, b], we express by writing f R[a, b] or R simply.

Outer and Inner Measure of a Bounded Set:

Let A [a, b] be any bounded subset of real numbers. The outer measure of A denoted by m*(A), is roughly defined as m*(A) = inf I(F), where the infimum is taken over all open sets F which contain A. F being open, can be expressed as a countable union of open intervals In, n = 1, 2, 3,……. Such that A.

For each such countable collection containing A, consider the sum of the lengths of the intervals in that collection. The outer measure of A is precisely then defined as

$$m^*(A)= A$$

The inner measure of A denoted by $m^*(A)$, is defined as, $m^*(A)=\sup m^*(B)$, where the supremum is taken over the outer measures of all the closed sets B contained in the set A.

A First Return Examination of the Lebesgue Integral:

The main purpose of this chapter is to prove the following theorem, which shows that such a procedure is, indeed, available and is closely akin to that of Riemann integration. This is the main result of the article.

Theorem: Suppose $f : \prod n \mapsto \mathbb{R}$ is a Lebesgue-integrable function. Then there is a countable dense set D inΠ^n and an enumeration $(x_p : p \in N)$ of D such that for each $\in> 0$ there is a $\delta> 0$ such that if P is a partition of Π^n having norm less than δ, then

$$|\sum_{j\in p} f(r(J))|J|- \int \Pi f|<\in,$$

Where (J) denotes the first element of the sequence that belongs to J. Before proving this result, we need to establish some notations and verify an elementary lemma which will be used repeatedly in the proof of the theorem.

Throughout this chapter the dimension n of our Euclidean space $\mathbb{R}^n$ is fired and denotes the unit "square" in r; that is, is the cartisian product of n copies of the unit interval [0,1].

We shall use to denote the Lebesgue n-dimensional measure of a measurable set A c R_n and shall use DS and so to denote the boundary and interior, respectively, of a set $S \subseteq \mathbb{R}^n$. By a "rectangle" we mean a set J of the form

$$J=[a_1,b_1]\times[a_2,b_2] \times \ldots\ldots\ldots\ldots \times[a_n, b_n],$$

Where each $a_i<b_{ij}$; we call each $[a_i,b_i]$ a side of J.

A partition Pof$\Pi_n$is a finite collection of non-overlapping rectangles whose union is In Π_n. (By non-overlapping, we mean that if $J_1\neq J_2$ belong to P,then$\lambda(J_1\cap J_2=0.)$ An elementary fact that we shall use in the proof of the lemma is that no point of Π_n belongs to more than 2^n rectangles $J \in P$. The norm of P, $||P||$, is the maximum of the lengths of the sides of all of the, $J \in P$.

Let $i \in N$ and for each j= 0, 1, ...2^i, let $c_j =j/2^i$. The uniform i- partition of Π^n, Q_i, is the collection of all rectangles of the form

$$[c_{j1},c_{j1+1}]\times[c_{j2},c_{j2+1}]\times\ldots\ldots\ldots\times[c_{jn},c_{jn+1}],$$

Where each integer j_k satisfies $0\ j_k< 2^i$. If $A \subseteq B \subset\Pi^n$,we say that A is i-fine in B provided that for each $J \in Q_i$ for which $J^0\cap B\neq \emptyset$,it follows that $J^0\cap A \neq \emptyset$.

Fix a$y_t\in F$ and fix a $J \in G$ containing y_t, if such a J exists. There can be atmost 2^n such J's containing y_t. Without loss of generality suppose that

$A \cap J$ has positive measure. Since $J \cap S_A= \emptyset$, if $I \in Q_i$ and $I \subseteq J$, then $I\cap A = \emptyset$. Thus, if $I \in$, Q_i satisfies $I\cap (J\cap A) \neq \emptyset$, then $I\cap\partial J \neq \emptyset$. However, there are atmost $B(n).(2i)^{n-1}$ such $I \in Q_i$ .Hence,

$$\lambda\ (A\cap\cup_{J\in G} J)<2^n.K.B(n).(2^i)^{n-1}1/(2i)^n$$

$$= K.B(n).2n\frac{K.B(n).2n}{2i}<\eta,$$

Completing the proof

Proof of Theorem:

For each $j \in\mathbb{N}$ we set: $A_j=\{x:j-1\leq|f(x)|<j\}$,Hence, and note that since f is integrable, the series $\sum_{j=1}^{\infty} j\lambda(A_j)$ converges. ltill be convenient to denote the tails of this series by $\varsigma_j=\sum_{k=j+1}^{\infty} k\lambda(A_k)$.

For each j we use Lusin's theorem repeatedly to obtain a sequence, $\{A_j^i\}$, of pairwise disjoint, perfect subsets of A_j such that $\lambda\{A_j^i\}=\frac{\lambda(Aj)}{2i}$and the restriction of f to A_j^i, f $|A_j^i$, is continuous. Thus, for each j we have $\lambda(A_j)=\sum_{i=1}^{\infty} \lambda\ (A_j^i)$.

Also, for each j we set

$B_j=\cup_{k=1}^{j} \cup \cup_{k=1}^{j} A_j^i$,$C_j=\cup_{k=j+1}^{\infty} A_k$, and $D_j=\cup_{k=1}^{j} \cup \frac{\infty}{i=j+1} A_k^i$, and note that $\lambda (B_j) + \lambda(C_j)+\lambda(D_j) = 1$.

Furthermore, we set $B_j^*= {}^{Bj}/_{Bj\ -\ 1}$,where we take $B_0= \emptyset$. Note that for each j, $f|_{Bj}$ is continuous and is in absolute value lessthan j.

For each $j \in\mathbb{N}$, apply Tietze's extension theorem to obtain $f_j$as a continuous ectensionof $f|_{Bj}$ to all of $\Pi^n$with $|f_j(x)| < j$ for all $x \in\Pi^n$. for each $j \in\mathbb{N}$ let $\in_j =\frac{1}{2j}$ and let δ_j be a positive number such that δ_j witnesses the Riemann integrability of f_j over Π^n with respect to $\in_j$; that is, if P s(J) denotes any point in J, then

$$|\sum_{J\in P} f j(s(J))|J|-\int \Pi_n f_j|<\in_j\ (1.1)$$

Our next goal is to inductively by stages define the sequence (xp : p $\in$ N). At stage 1, we choose a finite set $S \subset B_l$ so that S is 1-fine in B_l. We list these points in any order as $x_1,x_2,\ldots.x_{p1}$. Now, suppose stage j has been completed with $x_1,x_2,\ldots.x_{pj}$having been selected and ordered. We proceed to stage j + 1. First select a finite subset S_{j+l} C B^*_{j+l} such that S_{j+l} is (j + l)-fine in B^*_{j+l}. We are going to apply the blocking lemma j times, each time taking $\eta=\frac{1}{(j+1)2j+1.}$

Initially, apply the blocking lemma with F = S_{j+l} and A= B_j^* to determine a finite subset $S_j \subset B_j^*$ which satisfies the conclusion of that lemma. We may clearly assume that S_j is (j + l)-fine in and contains no X_p,$p \leq p_j$, since all of the sets A_k^i are perfect. Next, assume that

$$S_j \subset B_j^*, S_{j-1} \subset B_{j-1}^*, \ldots\ldots\ldots\ldots\ldots, S_{j-k} \subset B_{j-k}^*$$

have been selected for some $0 \leq k \leq j-2$. Apply the blocking lemma with F $=\bigcup_{i=-1}^{k} S_{j-i}$,A=B_{j-k-1}^*,to yield a finite set $S_{j-k-1} \subset B_{j-k-1}^*$. Again, we may assume that is (j + l)-fine in $B_{j-k-1}^*$and contains no x_p,$p \leq p_j$. We do this for each $0 \leq k \leq$ j-2. We now complete stage j +1 by appending the points from $\bigcup_{k=-1}^{j-1} S_{j-k}$ to ($x_1,x_2,\ldots\ldots x_{pj}$), first appending those from S_l(in any order), ……., and finally then those from $S_{j+1.}$ This completes stage j + 1 and we have defined $x_1,x_2,\ldots\ldots x_{pj},x_{pj+1},\ldots\ldots x_{pj+1}$.

Once all stages have been carried out, the sequence (xp : p e N) has been completely specified and it remains to show that this sequence accomplishes what the theorem claims. First, note that if D = $\{x_p : p \in \mathbb{N}\}$, then D is clearly dense Π^n in .

Before proceeding to see that the rest of the conclusion holds, we wish to make an additional observation. Fix a j $\in \mathbb{N}$ and let p be any partition of Π^nWe shall let B(n) denote the number of(n- l)-dimensional rectangles of (n - l)-dimensional measure one which form the boundary of Π^n. In proving the lemma we shall make use of the elementary fact that if J $\subseteq \Pi^n$ is any rectangle, then the number of elements of Q_i which intersect the boundary of J is atmost $B(n).(2^i)^{n-1}$.

A Random Approach to the Lebesgue Integral:

Introduction:

In this chapter we define an integral on Lebesgue measurable, real valued functions whose construction is similar to that of the Riemann integral. This is done by using Riemann sums which are random variables, and taking their limit in probability. This limit when it exists we call the random Riemann integral. The Riemann integral only exists for functions which are bounded and whose discontinuity points are a Lebesgue null set, the convergence of the random Riemann integral requires only a much weaker condition on the function to be integrated. We prove in fact that this integral exists and is equal to the Lebesgue integral if the Lebesgue integral exists.

We then prove further results on the convergence of Riemann sums treated as random variables, and its dependence on both the size of the function and those of the interval partitions on which the Riemann sums are constructed.

The idea of the random Riemann integral comes from that of the first return integral in the article , which is the base of this chapter. A sequence of real numbers which is dense in the unit interval determines an interval function; the first term of the sequence which belongs to the interval is the first return point of the interval. The first return integral was first suggested in , which was discussed in the previous chapter.

Notations:

In what follows, f is a Lebesgue measurable function from the unit interval Π:= [0, 1] into $\mathbb{R}$. We write the Lebesgue measure of a set $A \subset \mathbb{R}$ as |A|.

The Lebesgue integral of a function f on a set A is denoted $\int_A$ f and on Π simply by $\int$ f. A partition P is a finite collection of non degenerate intervals $\{I_k \subseteq \Pi : 1 \leq k \leq n\}$ such that the interiors of any two intervals are disjoint and the union of all the intervals is Π. The size of P is |P|:= max($|I_k|$: $I_k \in P$).

Random Riemann Sums:

Suppose f : $\Pi \longmapsto \mathbb{R}$ is a Lebsgue measurable function. Given a partition P, we define the random Riemann sum of f on P as follows.

For each $1_k \in$ P, let $t_k \in 1_k$ be a random variable with the uniform distri- bution on that interval, and with ti and tj independent for all i$\neq$ j. Then define random variables $X_k=|I_k|$. f (t_k) for each k. Note that E(f(t_k)) =$\frac{1}{|I_k|}$ $\int_{1k}$ f , and so E(X_k) =$\int_{1k}$f and further $E(X_k^p) = |I_k|^{p-1} . \int_{1k} f^p$ if this integral exists. The random Riemann sum of f on P is S_p (f):= $\sum X_k$. It is monotone and linear in f, and has expectation, E(S_p(f)) = $\sum \int_{Ik}$ f $= \int$ f.

This suggests that the random Riemann sum approximates the integral of a function. We hope to find a weak or strong law of large numbers which will allow us to define an integral from the random Riemann sum, equal to the Lebesgue integral. We will also use that, $E\sum |X_{k|^p} = \sum |I_{k|^{p-1}} . \int_{I_k} |f|^p . \leq |p|^{p-1} \int |f|^p$(3.1)

Almost Every Sequence Integrates:

Introduction and Notation:

We let $\Omega = \{\bar{x}: \mathbb{N} \rightarrow [0,1]\}$ denote the standard sequence space with the usual product measure μ. We shall use both x(p) and $x_p$to denote the p-th term of a sequence x. If F : $\Omega \rightarrow \mathbb{R}$ is μ-kintegrable, we denote its integral by E(F) = $\int_\Omega$ F$d\mu$. Also, if S is any proposition about sequences, P(S) = μ({x:S(x) is true}) denotes the probability that S is true.

We denote the Lebesgue measure of a (measurable) set A ⊆ℝ by λ(A) and the Lebesgue integral of a real-valued and integrable f over A by$\int_A$f; in the special case that A = [0, 1] we abbreviate this as∫ f. We use l(1) to denote the length of an interval.

Let f : [0, 1] → ℝ be Lebesgue integrable statement . For each interval, I ⊆ [0, 1], define $F_I : \Omega \to \mathbb{R}$ by F_1(x) = f or((x,I)) where r((x,I)) =x_p with p = min{n : x_p ∈ I} if such a minimum exists, or F_I(x) =-∞ if x(n) ∉ I

for all n ∈ ℕ. Let J ⊆ [0, 1] be an interval, let P denote a partition of J and ||P|| denote its mesh. We say that the sequence x ∈ *Ω* integrates f on J if, $\sum_{I\in P} F_{I\cap J}$(x)l(I∩J) =$\int_J$f. (3.1)

We say that integrates f if x integrates f on every interval J ⊆ [0, 1]. It was shown in chapter l that for each Lebesgue integrable f : [0, 1]→ ℝ, there is a sequence x which integrates f.

M.J .Evans and P.D.Humke are interested in determining the measure $\mu$of the set of sequences x which integrate a given Lebesgu eintegrable function f :[0,1]→ ℝ.This work is presented in this chapter. Throughout the chapter we concern ourselves with the case of a bounded measurable function f : [0, 1]→ ℝ. Suppose $\{p_n\}$ is a sequence of partitions of [0, 1] with mesh converging to 0. If for a sequence x ∈ *Ω*

$\sum_{I\in p_n} F_{I\cap J}$(x) l(I∩J) =$\int_J$f (3.2)

for each subinterval J ⊆ [0, 1], we say that x integrates f with respect to the sequence $\{P_n\}$. The following remark follows readily from the Lebesgue Density Theorem.

Theorem:

Suppose f : [0, 1] → ℝis bounded and measurable. Then there is a sequence of partitions $P_n$such that for almost every x ∈ *Ω*,x integrates f with respect to the sequence $\{P_n\}$.

The nature of the sequence Pn from Theorem 3.1 depends on the nature of the bounded measurable function f. The purpose of this chapter is to show that under rather general circumstances, any sequence of partitions will do. The condition we use is that the sequence of meshes $\{\|P_n\|\}$ is summable, or more generally that some power of the mesh sequence is summable.

Theorem:

Suppose f : [0, 1] →ℝ is bounded and measurable and for each n ∈ ℕ let P_n be a partition of [0, 1] with$\|P_n\|= m_n$. If $\{ m_n\}\in \bigcup_{j=1}^{\infty} lj$, then for almost every G o, j integrates f with respect to the sequence$\{P_n\}$.

Proof:

Let B > 0 be a bound for If). Fic a sequence of partitions $\{P_n\}$ and suppose that (3.2) is true for any fixed interval J ⊆ [0, 1] for a.e .sequence x ∈ *Ω* . Then, by intersecting countably many sets of full measure, it is easy to see that (3.2) is true for any countable collection of intervals for a.e. sequence x ∈ *Ω*. However, if (3.2) holds for all rational intervals and a.e. sequence x ∈ *Ω*, then almost every sequence integrates f with respect to $\{P_n\}$. Thus, it suffices to show that (3.2) is true for J = [0, 1] for a.e. sequence x ∈ Ω.

Define functions$f_{n,i} : \Omega\to \mathbb{R}$and $g_{n,i} : \Omega\to \mathbb{R}$as follows

$f_{n,i}$(x) =f or (x, $I_{n,i}$).l($I_{n,i}$) = $F_{I_{n,i}}$(x).l($I_{n,i}$), $g_{n,i}$(x) = $f_{n,i}$(x) -$a_{n,i}$where a_{n,i_i}= $\int_{I_{n,i}}$ f .

It follows that E ($f_{n,i}$) =$a_{n,i}$, so that E($g_{n,i}$), and an easy computation Shows that E($\int_{n,i}^k$)=($\int_{I_{n,i}}^k f^k).l^{k-1}(I_{n,i})$. Since both $|f_{n,i}|$and$|a_{n,i}|$are at most Bl($I_{n,i}$) we have $|g_{n,i}|$≤2Bl ($I_{n,i}$) and consequently $|E(g_{n,i}^k)|\leq 2^k B^k l^k(I_{n,i})$. (3.3)

By hypothesis, there is a p ∈ ℕ such that $\{m_n^p\}$ is summable. If $S_n=\sum_{i=1}^{M_n} g_{n,i}$, we next wish to show that $E(S_n^{2p}) = O(m_n^p)$ Let us adopt the notation,

K={$(k_1, k_2, \ldots \ldots k_{M_n})$ ∈{0,1,2,3,……,2_p}M_n : $\sum_{i=1}^{M_n} k_i = 2_p$ } and k* ={$(k_1, k_2, \ldots \ldots k_{M_n})$ ∈ K : k_i≠1 for each i=1,2,…….M_n}

Using the multinomial theorem, the linearity of expectation, and the independence of the functions $g_{n,i}$, we obtain

$$E(S_n^{2p}) = E((\textstyle\sum_{i=1}^{M_n} g_{n,i})^{2p})=\sum_{(k_1,k_2,\ldots\ldots k_{M_n})\in K}(\binom{2_p}{k_1,k_2,\ldots\ldots k_{M_n}})\prod_{i=1}^{M_n} E\,(g_{n,i}^{k_i}))$$

Then, using the fact that each E($g_{n,i}$) = 0, we have $E(S_n^{2p}) =\sum_{(k_1,k_2,\ldots\ldots k_{M_n})\in K}(\binom{2_p}{k_1,k_2,\ldots\ldots k_{M_n}})\prod_{i=1}^{M_n} E\,(g_{n,i}^{k_i}))$

From this and inequality (3.3) we obtain

$$|\, E(S_n^{2p})| \leq \sum_{(k_1,k_2,\ldots\ldots k_{M_n})\in K}(\binom{2_p}{k_1,k_2,\ldots\ldots k_{M_n}})\prod_{i=1}^{M_n} E\,(g_{n,i}^{k_i})|)$$

$$\leq (2_p)!\sum_{(k_1,k_2,\ldots\ldots k_{M_n})\in K}(\prod_{i=1}^{M_n} E\,(g_{n,i}^{k_i})|) \leq (2_p)!\sum_{(k_1,k_2,\ldots\ldots k_{M_n})\in K}(\prod_{i=1}^{M_n} E\,(g_{n,i}^{k_i}))$$

$$=(2p)!2^{2p}B^{2p}\sum_{(k_1,k_2,\ldots\ldots k_{M_n})\in K}(\prod_{i=1}^{M_n} l^{k_i}\,(l_{n,i})\ (3.4)$$ *N*ext, we observe that

$$\sum_{(k_1,k_2,\ldots\ldots k_{M_n})\in K}(\prod_{i=1}^{M_n} l^{k_i}(I_{n,i})) \leq \sum_{i_p=1}^{M_n}\ldots\ldots\sum_{i_2=1}^{M_n}\sum_{i_1=1}^{M_n}(l^2(I_{n,i_1})]^2(I_{n,i_1})\ldots\ldots l2(I_{n,i_p}))$$

$$= \sum_{i_p=1}^{M_n}(l^2(I_{n,i_p})(\ldots.(\sum_{i_2=1}^{M_n}(l^2(I_{n,i_2})(\sum_{i_1=1}^{M_n} l^2(I_{n,i_1}))))\ldots.)) \leq m_n^p,\ (3.5)$$

Where the final inequality follows from the observation that for each i, $\sum_{i=1}^{M_n} l^2(g_{n,i}) \leq m_n \sum_{i=1}^{m_n} l\,(I_{n,i}) = m_n$.

From (3.4) and (3.5) we obtain the bound $|E(S_n^{2p})| \leq (2p)!4^p B^{2p} m_n^p$.

Now, let c > 0 be given. We wish to compute P($|\sum_{i=1}^{M_n} f$ or ($\bar{x}$,I_{n_i})l(I_{n_i})- $\int f| \geq \in$).
We have, P($|\sum_{i=1}^{M_n} f$ or ($\bar{x}$,I_{n_i})l(I_{n_i})- $\int f| \geq \in$) = p($|\sum_{i=1}^{M_n} g_{n,i}| \geq \in$) = P($|s_n| \geq \in$).
However, P($|s_n| \geq \in$) $\leq \frac{E(S_n^{2p})}{e^{2p}} \leq \frac{(2p)!.4^p.B^{2p}}{\in^{2p}} m_n^p$.

Since $\{m_n^p\}$ is summable, it follows from the Borel-Cantelli Lemma that limsup A_n has measure zero where $A_n = \{\bar{x}:|Sn(\overline{x})| \geq \in\}$. It is easy to see that if $\bar{x} \notin$ limsup A_n, then $\bar{x}$ integrates f with respect to the sequence $\{P_n\}$ so this completes the proof.

Conclusion:

This section presents functions commonly used in this role, but not the verifications that these functions have the required properties. The Basic Theorem and its Corollary are often used to simplify proofs. Counting as a process of determinantthe numbers of elements of a finite set of objects. Always traditional way of counting consists as continually increasing a counter by a unit for every element of the set, in some order, while marking those elements to avoid viewing the same absolute element more than once, up to that no unmarked elements are left; if the counter was set to one after the first object, the value after analysing the final object gives the desired number of elements. The relative terminology enumeration refers to a uniq identifying the elements of the finite set or a infinite set by assigning a number to each element.

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